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Classically integrable boundary conditions for affine Toda field theories

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Abstract

Boundary conditions compatible with classical integrability are studied both directly, using an approach based on the explicit construction of conserved quantities, and indirectly by first developing a generalisation of the Lax pair idea. The latter approach is closer to the spirit of earlier work by Sklyanin and yields a complete set of conjectures for permissible boundary conditions for any affine Toda field theory.

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1. Introduction

Besides its classical integrability [1-3], affine Toda field theory appears to be quantum integrable and, over recent years, much has been discovered concerning the spectrum and scattering in the real coupling regime [4-9]. The classical integrability of the theory stems from the existence of a Lax pair representation of the field equations, leading to an infinite set of independent conserved charges in involution.

More recently, there has been some interest in examining the Toda models on a half-line or on a finite interval. In particular, certain special solutions to appropriately modified bootstrap relations have been obtained by Fring and Köberle [10] and by Sasaki [11] but without establishing the precise nature of the boundary conditions responsible for them. For the special case of the sine (or sinh)-Gordon model, there is already a substantial literature, examining the classically permissible boundary conditions which preserve integrability and calculating the effects of the allowable boundaries in quantum field theory [12-18]. It appears [15] the most general boundary condition (at $x^1 = 0$, say) is of the form:

$$\frac{\partial \phi}{\partial x^1} = \frac{a}{\beta} \sin \beta \left(\frac{\phi - \phi_0}{2} \right) \qquad \text{at} \qquad x^1 = 0, \tag{1.1}$$

where a and ϕ_0 are arbitrary constants, and β is the sine-Gordon coupling. Some years ago, Sklyanin and others [13,14] have argued for restricted versions of (1.1) in the classical theory, and MacIntyre [18] has recently demonstrated that (1.1) preserves classical integrability. Ghoshal and Zamolodchikov [15,17] have also presented arguments supporting the idea that the quantum sine-Gordon theory with a boundary does indeed have a pair of additional coupling constants associated with the boundary, although it is not yet completely clear in what way these are related to the parameters a and ϕ_0 appearing in (1.1) (but see also [16] for further developments in this direction).

In an attempt to explain the variety of solutions exhibited in [10,11], the classical charges of affine Toda theory were studied in [19,20]. Surprisingly, it was discovered that the possible boundary conditions for the $a_n^{(1)}$ and $d_n^{(1)}$ theories are highly constrained by the requirement that there should be conserved modifications of the spin two or three charges even in the presence of the boundary. Effectively, in those cases, there is only a discrete ambiguity and the possible boundary conditions are summarised by adding a term to the action[1] of the form

$$\mathcal{L}_{\text{boundary}} = -\delta(x^1)\mathcal{B}(\phi), \tag{1.2}$$

[1] The notation and conventions for affine Toda field theory are those of [5]

where

$$\mathcal{B}=\frac{m}{\beta^2}\sum_0^r A_i e^{\frac{\beta}{2}\alpha_i\cdot\phi}, \tag{1.3}$$

and the coefficients $A_i,\ i=0,\dots,r$ are a set of real numbers with

$$\textbf{either } |A_i|=2\sqrt{n_i},\ \text{ for } i=0,\dots,r \textbf{ or } A_i=0 \text{ for } i=0,\dots,r\ . \tag{1.4}$$

In this paper, these arguments will be elaborated and extended to include spin four charges (thereby encompassing the the case $e_6^{(1)}$).

The constraints on the boundary term are seen to be necessary, following the arguments of [19,20] and below, but not sufficient; there is always the chance that a study of charges with spins greater than three or four may lead to stronger constraints on the coefficients A_i appearing in eq(1.3). With this worry in mind, it is clearly desirable to find an alternative approach to the classical integrability, preferably one which is close to the Lax pair idea, even in the presence of boundary conditions at one specific value of x^1. (Or possibly two, if the theory is defined on an interval.)

One of the purposes of this article is to report on a definition of the Lax pair for affine Toda theory which successfully incorporates as a consequence of the zero curvature requirement not only the equations of motion but also the boundary conditions. The Lax pair argument provides the missing sufficiency requirement for the boundary condition restrictions and permits conjectures to be made for possible boundary conditions in those cases for which the direct approach in terms of charges is not tractable.

One corollary of being able to do this is the discovery that although the form (1.3) appears to be universal, the further constraints (1.4) are not universal. In fact, the stringent constraints on the coefficients A_i are peculiar to the simply-laced Toda field theories. The non-simply laced theories are curiously different; for them, some of the coefficients can be chosen freely[2]. The $a_2^{(2)}$ example illustrates this. An examination of the spin five charge in the presence of a boundary condition at $x^1=0$ leads to a boundary condition of the form (1.3), ie

$$\mathcal{B}=A_1e^{\phi}+A_0e^{-\phi/2},$$

[2] These observations may be connected with the remarkable duality properties of quantum affine Toda field theory on the full line: the simply-laced data (and $a_{2n}^{(2)}$) lead to self-dual field theories while the non-simply-laced data (except $a_{2n}^{(2)}$) are really dual pairs [9].

and the further constraint

$$A_0(A_1^2-2)=0.$$

The sine-Gordon theory remains the only example within the set of Toda field theories for which integrability dictates the form of the boundary condition but places no further constraints on the the boundary coefficients[3].

2. Conserved charges on the whole and half-line

The usual lagrangian density for the full line Toda theory is

$$\mathcal{L}_T=\frac{1}{2}\partial_\mu\phi^a\partial^\mu\phi^a-V(\phi) \tag{2.1}$$

where

$$V(\phi)=\frac{m^2}{\beta^2}\sum_0^r n_i e^{\beta\alpha_i\cdot\phi}. \tag{2.2}$$

(For the classical discussion, the coupling constant β and mass scale m will be discarded from now on.)

On a half line, $x^1<a$, say, with a boundary condition at $x^1=a$, the classical lagrangian is effectively replaced by

$$\mathcal{L}=\theta(-x^1+a)\mathcal{L}_T-\delta(x^1-a)\mathcal{B}, \tag{2.3}$$

where for the purposes of the present discussion $\mathcal{B}$ is a function of the field only, not its derivatives. The lagrangian $\mathcal{L}$ leads to the field equations on $x^1<a$ and to the boundary condition

$$\partial_1\phi=-\frac{\partial\mathcal{B}}{\partial\phi}, \tag{2.4}$$

at $x^1=a$.

The classical conserved charges Q_s of affine Toda field theory on the full line can be calculated in principle using a Lax pair. Later, a generalisation of that procedure to the half-line will be given but, in this section, the spin 2, 3 and 4 charges for the simply-laced cases both on the whole and the half-line will be calculated in a pedestrian fashion, following [19,20].

[3] See also the discussion of classical sinh-Gordon in [20].

On the full line, a density $T_{\pm(s+1)}$ defines a conserved charge $Q_{\pm s}$ if it satisfies (using the equations of motion)

$$\partial_{\mp} T_{\pm(s+1)} = \partial_{\pm} \Theta_{\pm(s-1)} \tag{2.5}$$

for some $\Theta_{\pm(s-1)}$. The conserved charge is then given by

$$Q_{\pm s} = \int_{-\infty}^{+\infty} dx^1 (T_{\pm(s+1)} - \Theta_{\pm(s-1)}). \tag{2.6}$$

For an affine Toda theory on the full line based on an algebra g there is an infinite set of conserved charges, one for each integer s of the form $s = m + nh$, where m is an exponent of the algebra g, n is an integer, and h is its Coxeter number.

In order for the theory on the half-line to remain integrable, an infinite set of conserved charges should continue to exist. Clearly, translational invariance is destroyed and the only combination of spin ± 1 charges which is left in the theory restricted to a half line is the energy. For any boundary condition this is given by

$$E = \hat{Q}_1 + \hat{Q}_{-1} + \mathcal{B}, \tag{2.7}$$

where hatted quantities are the standard densities integrated over the half line. For the higher spins considered here it is found there are conserved quantities of the form

$$P_s = \hat{Q}_s + \hat{Q}_{-s} - \Sigma_s, \tag{2.8}$$

where the boundary term Σ_s is defined by the condition

$$(T_{s+1} + \Theta_{s-1} - T_{-s-1} - \Theta_{-s+1}) = \partial_0 \Sigma_s. \tag{2.9}$$

Such conserved quantities will be referred to as 'spin s'.

For $s = 1$, condition (2.8) is automatically fulfilled by choosing $\Sigma_1 = -\mathcal{B}$. However, for the higher spin charges (2.9) severely restricts the form of the boundary conditions. It will be shown that $\mathcal{B}$ must take the form (1.3) where the coefficients $A_i, i = 0, \ldots, n$ are a set of real numbers. Furthermore, conservation of the charges considered here restricts the values of the A_i in the cases $e_6^{(1)}, d_n^{(1)}, a_n^{(1)}, n \geq 2$ to those satisfying (1.4).

2.1. Spin two charges

A general ansatz for $T_{\pm 3}$ (using light-cone coordinates $x^{\pm} = (x^0 \pm x^1)/\sqrt{2}$) reads

$$T_{\pm 3} = \frac{1}{3} A_{abc} \partial_{\pm}\phi_a \partial_{\pm}\phi_b \partial_{\pm}\phi_c + B_{ab} \partial_{\pm}^2 \phi_a \partial_{\pm}\phi_b, \tag{2.10}$$

where the coefficients A_{abc} are completely symmetric and the coefficients B_{ab} are antisymmetric. An explicit calculation reveals that (2.5) is satisfied for

$$\Theta_{\pm 1} = -\frac{1}{2} B_{ab} \partial_{\pm}\phi_a V_b, \tag{2.11}$$

provided that

$$A_{abc} V_a + B_{ab} V_{ac} + B_{ac} V_{ab} = 0, \tag{2.12}$$

where

$$V_b = \frac{\partial V}{\partial \phi_b}, \qquad V_{bc} = \frac{\partial^2 V}{\partial \phi_b \partial \phi_c}, \qquad \text{etc.} \tag{2.13}$$

For practical calculations it is convenient to introduce the notation

$$A_{ijk} = A_{abc} (\alpha_i)_a (\alpha_j)_b (\alpha_k)_c, \qquad B_{ij} = B_{ab} (\alpha_i)_a (\alpha_j)_b, \tag{2.14}$$

and

$$C_{ij} = \alpha_i \cdot \alpha_j .$$

Then eq(2.12) implies

$$A_{ijk} + B_{ij} C_{ik} + B_{ik} C_{ij} = 0. \tag{2.15}$$

This equation is very restrictive and fixes both A_{ijk} and B_{ij} up to an overall constant. In fact, B_{ij} is non-zero only for the $a_n^{(1)}$ cases and, for those cases (and $n > 1$), $B_{ij} = 0$ except for $j = i \pm 1 \mod n+1$, and $B_{i-1\, i} = B_{i\, i+1},\ i = 1, \ldots, n+1$. The sets of coefficients A_{abc} and B_{ab} are found by inverting the transformations (2.14) and lead to the following conserved current densities for $a_n^{(1)}$ $(n > 1)$:

$$T_{\pm 3}^{(n)} = \frac{2(2i)}{3\sqrt{n+1}} \delta_{a+b+c, 0 \bmod (n+1)} \partial_{\pm}\phi_a \partial_{\pm}\phi_b \partial_{\pm}\phi_c + \Gamma_2^a \delta_{a+b, n+1} \partial_{\pm}^2 \phi_a \partial_{\pm}\phi_b, \tag{2.16}$$

where

$$\Gamma_s^a = \frac{\sin\left(\frac{s a \pi}{h}\right)}{\sin^s\left(\frac{a\pi}{h}\right)}, \tag{2.17}$$

with h the Coxeter number of the algebra.

On the half-line, a spin two charge P_2 exists if condition (2.9) is satisfied for some Σ_2. Substituting the definitions (2.10) and (2.11) in (2.9) for $s = 2$ it is found that Σ_2 does not exist unless the following two conditions hold at $x^1 = 0$:

$$A_{abc}\mathcal{B}_a + 2B_{ab}\mathcal{B}_{ac} + 2B_{ac}\mathcal{B}_{ab} = 0, \tag{2.18}$$

$$\frac{1}{3}A_{abc}\mathcal{B}_a\mathcal{B}_b\mathcal{B}_c + 2B_{ab}V_a\mathcal{B}_b = 0. \tag{2.19}$$

Similar notation to that of (2.13) is being used for derivatives of $\mathcal{B}$. Once conditions (2.18), (2.19) are solved the extra piece in eq(2.8) will be given by

$$\Sigma_2 = -\sqrt{2}B_{ab}\partial_0\phi_a\mathcal{B}_b. \tag{2.20}$$

Both conditions involve the boundary term $\mathcal{B}$. Comparing (2.18) with (2.12) reveals that $\mathcal{B}$ must be equal to

$$\sum_0^r A_i e^{\alpha_i\cdot\phi/2},$$

apart from an additive arbitrary constant. The second condition, eq(2.19), is nonlinear in the boundary term and therefore provides equations for the constant coefficients A_i in terms of the coefficients in the potential. In one way of analysing these equations, the explicit expressions for A_{abc} and B_{ab}, which for $a_n^{(1)}$ are defined by (2.16), are substituted to find that the A_i have to satisfy (2.4).

2.2. Spin three charges

In this case, the following ansatz for the conserved current is appropriate:

$$T_{\pm 4} = \frac{1}{4}A_{abcd}\partial_\pm\phi_a\partial_\pm\phi_b\partial_\pm\phi_c\partial_\pm\phi_d + \frac{1}{2}B_{abc}\partial_\pm^2\phi_a\partial_\pm\phi_b\partial_\pm\phi_c + \frac{1}{2}D_{ab}\partial_\pm^3\phi_a\partial_\pm\phi_b, \tag{2.21}$$

where A_{abcd} and D_{ab} are completely symmetric. The coefficients B_{abc} are symmetric in their last two indices but are ambiguous up to a totally symmetric part; adding a totally symmetric part to B_{abc} will only add a total $\partial_\pm$ derivative to $T_{\pm 4}$, which will not change condition (2.5).

The above expression corresponds to a conserved charge on the whole line if

$$\begin{aligned} &B_{[ab]c}V_c - D_{c[a}V_{b]c} = 0 \\ &A_{abcd}V_a + \frac{1}{2}B_{a(bc}V_{d)a} - \frac{1}{2}V_{a(d}B_{bc)a} + \frac{1}{2}D_{a(b}V_{cd)a} = 0. \end{aligned} \tag{2.22}$$

Then

$$\Theta_{\pm 2} = -\frac{1}{4} B_{abc} V_b \partial_\pm \phi_a \partial_\pm \phi_c - \frac{1}{4} D_{ab} V_a \partial_\pm^2 \phi_b. \tag{2.23}$$

Here it has been convenient to introduce a bracket notation

$$\begin{aligned} M_{(a_1 \cdots a_n)} &= \frac{1}{n!} \sum_{\{\text{permutations}\}} M_{p(a_1) \cdots p(a_n)}, \\ M_{[a_1 \cdots a_n]} &= \frac{1}{n!} \sum_{\{\text{permutations}\}} \text{sign}\{p\} M_{p(a_1) \cdots p(a_n)}. \end{aligned} \tag{2.24}$$

Eqs(2.22) have solutions for both $a_n^{(1)}$ and $d_n^{(1)}$. They have been computed using Mathematica after transforming the equation into a form similar to (2.15). The expressions for $T_{\pm 4}$ which are found this way can be written in a nice form by choosing the completely symmetric part of B_{abc} conveniently. For $a_n^{(1)}$ the expressions are

$$\begin{aligned} T_{\pm 4}^{(n)} =& \frac{2(2i)^2}{4(n+1)} \left\{ \delta_{a+b+c+d,0 \text{ mod } (n+1)} - \frac{3}{2} \delta_{a+b,n+1} \delta_{c+d,n+1} \right\} \\ & \qquad \times \partial_\pm \phi_a \partial_\pm \phi_b \partial_\pm \phi_c \partial_\pm \phi_d \\ &+ \frac{3(2i)\Gamma_2^a}{2\sqrt{n+1}} \delta_{a+b+c,0 \text{ mod } (n+1)} \partial_\pm^2 \phi_a \partial_\pm \phi_b \partial_\pm \phi_c \\ &+ \Gamma_3^a \delta_{a+b,0 \text{ mod } (n+1)} \partial_\pm^3 \phi_a \partial_\pm \phi_b, \end{aligned} \tag{2.25}$$

while for $d_n^{(1)}$

$$\begin{aligned} T_{\pm 4}^{(n)} =& - \frac{1}{(n-1)} \left\{ \delta_{a+b+c+d,2(n-1)} + 3\delta_{a+b-c-d,0} - 4\delta_{-a+b+c+d,0 \text{ mod } 2(n-1)} - 3\delta_{a,b}\delta_{c,d} \right\} \\ & \qquad \times \partial_\pm \phi_a \partial_\pm \phi_b \partial_\pm \phi_c \partial_\pm \phi_d \\ &+ \frac{3\Gamma_2^a}{\sqrt{2(n-1)}} \Big[\left\{ \delta_{a+b+c,2(n-1)} - 2\delta_{a-b+c,0} + \delta_{-a+b+c,0} \right\} \partial_\pm^2 \phi_a \partial_\pm \phi_b \partial_\pm \phi_c \\ &- \left[1 + (-)^{n+a} \right] \left\{ (\partial_\pm \phi_s)^2 + (\partial_\pm \phi_{s'})^2 \right\} \partial_\pm^2 \phi_a - 2 \left[1 - (-)^{n+a} \right] \partial_\pm \phi_s \partial_\pm \phi_{s'} \partial_\pm^2 \phi_a \Big] \\ &+ \Gamma_3^a \delta_{a,b} \partial_\pm^3 \phi_a \partial_\pm \phi_b - 4\Delta_{\pm 4}^{(n)}, \end{aligned} \tag{2.26}$$

with

$$\begin{aligned} \Delta_{\pm 4}^{(n)} =& \left\{ \partial_\pm \phi_s \partial_\pm^3 \phi_s + \partial_\pm \phi_{s'} \partial_\pm^3 \phi_{s'} \right\} - \frac{3}{2(n-1)} \left\{ (\partial_\pm \phi_s)^2 + (\partial_\pm \phi_{s'})^2 \right\} (\partial_\pm \phi_a)^2 \\ &+ \frac{(n-4)}{4(n-1)} \left\{ (\partial_\pm \phi_s)^4 + (\partial_\pm \phi_{s'})^4 \right\} + \frac{3(n-2)}{2(n-1)} (\partial_\pm \phi_s)^2 (\partial_\pm \phi_{s'})^2 \end{aligned} \tag{2.27}$$

for even n, and

$$\Delta_{\pm 4}^{(n)} = \left\{ \partial_\pm \phi_s \partial_\pm^3 \phi_{\bar{s}} + \partial_\pm \phi_{\bar{s}} \partial_\pm^3 \phi_s \right\} - \frac{3}{(n-1)} \partial_\pm \phi_s \partial_\pm \phi_{\bar{s}} \left(\partial_\pm \phi_a \right)^2 \\ + \frac{1}{4} \left\{ \left(\partial_\pm \phi_s \right)^4 + \left(\partial_\pm \phi_{\bar{s}} \right)^4 \right\} + \frac{3(n-3)}{2(n-1)} \left(\partial_\pm \phi_s \right)^2 \left(\partial_\pm \phi_{\bar{s}} \right)^2 \tag{2.28}$$

for odd n. For $d_4^{(1)}$ there is an extra spin 3 charge which is defined by

$$T_{\pm 4}^{(4)} = \frac{1}{2} \left\{ \left(\partial_\pm \phi_1 \right)^2 - \left(\partial_\pm \phi_2 \right)^2 \right\} \left\{ \left(\partial_\pm \phi_s \right)^2 - \left(\partial_\pm \phi_{s'} \right)^2 \right\} + \left\{ \partial_\pm \phi_s \partial_\pm^3 \phi_s - \partial_\pm \phi_{s'} \partial_\pm^3 \phi_{s'} \right\} \\ - \sqrt{2} \left\{ \partial_\pm^2 \phi_s \left[\partial_\pm \phi_1 \partial_\pm \phi_{s'} + \partial_\pm \phi_2 \partial_\pm \phi_s \right] - \partial_\pm^2 \phi_{s'} \left[\partial_\pm \phi_1 \partial_\pm \phi_s + \partial_\pm \phi_2 \partial_\pm \phi_{s'} \right] \right\} . \tag{2.29}$$

The conditions for a spin 3 charge P_3 to exist on the half-line are given by (2.9) for $s = 3$. One finds that at $x_1 = 0$ the following equations have to be satisfied:

$$B_{[ab]c} \mathcal{B}_c - 2 D_{c[a} \mathcal{B}_{b]c} = 0, \\ A_{abcd} \mathcal{B}_a + B_{a(bc} \mathcal{B}_{d)a} - \mathcal{B}_{a(d} B_{bc)a} + 2 D_{a(b} \mathcal{B}_{cd)a} = 0, \\ - \frac{1}{2} \left(A_{abcd} \mathcal{B}_a \mathcal{B}_b \mathcal{B}_c + B_{abc} \mathcal{B}_{ad} \mathcal{B}_b \mathcal{B}_c + B_{abd} V_a \mathcal{B}_b - B_{(db)a} V_a \mathcal{B}_b \right. \\ \left. + D_{a(d} V_{b)a} \mathcal{B}_b - 2 D_{ab} V_a \mathcal{B}_{bd} \right) = \frac{\partial \Sigma_3^{(0)}}{\partial \phi_d}, \tag{2.30}$$

and an expression for $\Sigma_3^{(0)}$ determined. Once the latter is found, the additional piece in the conserved quantity is given by

$$\Sigma_3 = -\frac{1}{2} B_{abc} \partial_0 \phi_a \partial_0 \phi_b \mathcal{B}_c - D_{ab} \mathcal{B}_a \partial_0^2 \phi_b - \frac{1}{2} D_{ab} V_a \mathcal{B}_b + \Sigma_3^{(0)}. \tag{2.31}$$

The first two of conditions (2.30) are automatically satisfied by the general boundary term $\mathcal{B}$ given in (1.3) as a consequence of (2.22). The last condition is non-linear in the boundary term and gives conditions on the parameters A_i. For $a_n^{(1)}$ these are the same conditions as those found from the spin two charge. The existence of a conserved spin three charge also restricts the parameters for $d_n^{(1)}$ to satisfy (1.4). Then $\Sigma_3^{(0)}$ is given by

$$\Sigma_3^{(0)} = \sum_{i,j=0}^{n} A_i^2 A_j \left[\frac{1}{6} \delta_{ij} - \frac{1}{4} \mathcal{A}_{ij} \right] e^{(2\alpha_i + \alpha_j) \cdot \phi / 2} \tag{2.32}$$

for $a_n^{(1)}$ and

$$\Sigma_3^{(0)} = \sum_{i,j=0}^{n} \left\{ A_i^2 A_j \left[-\frac{1}{24}\delta_{ij} + \frac{1}{16}\mathcal{A}_{ij} + \frac{3}{16}\left(\delta_{i,0}\delta_{j,1} + \delta_{i,1}\delta_{j,0} + \delta_{i,n-1}\delta_{j,n} + \delta_{i,n}\delta_{j,n-1}\right) \right.\right.$$
$$\left.\left. - \frac{1}{16}\left(\delta_{i,0}\delta_{j,0} + \delta_{i,1}\delta_{j,1} + \delta_{i,n-1}\delta_{j,n-1} + \delta_{i,n}\delta_{j,n}\right)\right] e^{(2\alpha_i+\alpha_j)\cdot\phi/2} \right\}$$
$$- \frac{3}{8}\left\{ A_0A_1A_2 e^{(\alpha_0+\alpha_1+\alpha_2)\cdot\phi/2} + A_{n-2}A_{n-1}A_n e^{(\alpha_{n-2}+\alpha_{n-1}+\alpha_n)\cdot\phi/2} \right\} \tag{2.33}$$

for $d_n^{(1)}$. Here $\mathcal{A}^{ij}$ is the adjacency matrix $2\delta_{ij} - C^{ij}$. For the extra spin 3 charge of $d_4^{(1)}$ which is defined by (2.29), $\Sigma_3^{(0)}$ vanishes.

2.3. Spin four charges

The most general ansatz for $T_{\pm5}$ reads

$$\begin{aligned} T_{\pm5} =& \frac{1}{5}A_{abcde}\partial_\pm\phi_a\partial_\pm\phi_b\partial_\pm\phi_c\partial_\pm\phi_d\partial_\pm\phi_e + \frac{1}{3}B_{abcd}\partial_\pm^2\phi_a\partial_\pm\phi_b\partial_\pm\phi_c\partial_\pm\phi_d \\ &+ \frac{1}{2}D_{abc}\partial_\pm^2\phi_a\partial_\pm^2\phi_b\partial_\pm\phi_c + E_{ab}\partial_\pm^4\phi_a\partial_\pm\phi_b, \end{aligned} \tag{2.34}$$

with A_{abcde} is completely symmetric, B_{abcd} symmetric in all but its first index and only defined modulo a totally symmetric part. The set of coefficients D_{abc} is symmetric in its first two indices and E_{ab} is anti-symmetric. The above ansatz for $T_{\pm5}$ corresponds to a conserved charge on the whole line under the following constraints

$$\begin{aligned} & D_{abc}V_c - 4E_{c(a}V_{b)c} = 0, \\ & B_{bcda}V_a - B_{(cd)ab}V_a + D_{ab(c}V_{d)a} - \frac{1}{2}V_{a(c}D_{d)ab} - \frac{1}{2}D_{a(cd)}V_{ab} \\ & \qquad - 2E_{ab}V_{acd} + 2E_{a(c}V_{d)ab} = 0, \\ & A_{abcde}V_a + \frac{1}{3}B_{a(bcd}V_{e)a} - \frac{1}{3}V_{a(e}B_{bcd)a} - \frac{1}{3}D_{a(bc}V_{de)a} + \frac{2}{3}E_{a(b}V_{cde)a} = 0. \end{aligned} \tag{2.35}$$

Then $\Theta_{\pm3}$ is given by

$$\begin{aligned} \Theta_{\pm3} =& -\frac{1}{6}\left\{B_{bcda}V_a + D_{abc}V_{ad} + E_{ab}V_{acd}\right\}\partial_\pm\phi_b\partial_\pm\phi_c\partial_\pm\phi_d \\ &+ \frac{1}{2}\left\{E_{ab}V_{bc} + E_{cb}V_{ba}\right\}\partial_\pm^2\phi_a\partial_\pm\phi_c - \frac{1}{2}E_{ab}V_b\partial_\pm^3\phi_a. \end{aligned} \tag{2.36}$$

Eqs (2.35) only have non-trivial solutions for $a_n^{(1)}$, $d_5^{(1)}$ and $e_6^{(1)}$. For $a_n^{(1)}$ these define the following conserved currents

$$\begin{aligned}
T^{(1)}_{\pm 5} =& \frac{2(2i)^3}{5(n+1)^{\frac{3}{2}}} \left\{\delta_{a+b+c+d+e,0 \bmod (n+1)} - (10/3)\delta_{a+b,n+1}\delta_{c+d+e,0 \bmod (n+1)}\right\} \\
&\qquad\qquad\qquad\qquad \times \partial_\pm\phi_a\partial_\pm\phi_b\partial_\pm\phi_c\partial_\pm\phi_d\partial_\pm\phi_e \\
&+ \frac{(2i)^2}{(n+1)}\Bigg[(4/3)\Gamma_2^a\left[\delta_{a+b+c+d,0 \bmod (n+1)} - (3/2)\delta_{a+b,n+1}\delta_{c+d,n+1}\right] \\
&\qquad + \Gamma_2^{a+b}\left[\delta_{a+b+c+d,0 \bmod (n+1)} - \delta_{a+b,n+1}\delta_{c+d,n+1}\right]\Bigg] \\
&\qquad\qquad\qquad\qquad \times \partial^2_\pm\phi_a\partial_\pm\phi_b\partial_\pm\phi_c\partial_\pm\phi_d \\
&+ \frac{(2i)}{(n+1)^{\frac{1}{2}}}\left\{-\frac{4}{3}\Gamma_3^a - \frac{4}{3}\Gamma_3^b + \frac{4}{3} + \Gamma_2^a\Gamma_2^b\right\}\delta_{a+b+c,0 \bmod (n+1)}\partial^2_\pm\phi_a\partial^2_\pm\phi_b\partial_\pm\phi_c \\
&+ \Gamma_4^a\delta_{a+b,n+1}\partial^4_\pm\phi_a\partial_\pm\phi_b.
\end{aligned} \tag{2.37}$$

For $d_5^{(1)}$ and $e_6^{(1)}$ the expressions for $T_{\pm 5}$ are quite long. Appendix A contains further details on these cases.

For a spin four charge to exist on the half line the condition (2.9) must hold. This requirement leads to the conditions

$$\begin{aligned}
&D_{abc}\mathcal{B}_c - 8E_{c(a}\mathcal{B}_{b)c} = 0, \\
&B_{bcda}\mathcal{B}_a - B_{(cd)ab}\mathcal{B}_a + 2D_{ab(c}\mathcal{B}_{d)a} - \mathcal{B}_{a(c}D_{d)ab} - D_{a(cd)}\mathcal{B}_{ab} \\
&\qquad - 8E_{ab}\mathcal{B}_{acd} + 8E_{a(c}\mathcal{B}_{d)ab} = 0, \\
&A_{abcde}\mathcal{B}_a - \frac{2}{3}\mathcal{B}_{a(e}B_{bcd)a} + \frac{2}{3}B_{a(bcd}\mathcal{B}_{e)a} - \frac{4}{3}D_{a(bc}\mathcal{B}_{de)a} + \frac{16}{3}E_{a(b}\mathcal{B}_{cde)a} = 0,
\end{aligned} \tag{2.38}$$

and

$$\begin{aligned}
&A_{abcde}\mathcal{B}_c\mathcal{B}_d\mathcal{B}_e - \mathcal{B}_{c(b}B_{a)cde}\mathcal{B}_d\mathcal{B}_e + B_{cde(a}\mathcal{B}_{b)c}\mathcal{B}_d\mathcal{B}_e + D_{cde}\mathcal{B}_{ac}\mathcal{B}_{bd}\mathcal{B}_e \\
&\qquad + \frac{1}{2}B_{cdab}V_c\mathcal{B}_d - \frac{1}{3}B_{(ab)cd}V_c\mathcal{B}_d - \frac{1}{6}B_{dabc}V_c\mathcal{B}_d - \frac{7}{6}V_{d(b}D_{a)dc}\mathcal{B}_c \\
&\qquad - \mathcal{B}_{c(a}D_{b)dc}V_d + D_{cd(a}\mathcal{B}_{b)d}V_c - 6E_{cd}V_{c(a}\mathcal{B}_{b)d} + \frac{8}{3}E_{d(a}V_{b)cd}\mathcal{B}_c \\
&\qquad + 2E_{d(a}\mathcal{B}_{b)c}V_{cd} - \frac{1}{6}D_{cd(a}V_{b)c}\mathcal{B}_d + \frac{1}{3}E_{cd}V_{abc}\mathcal{B}_d - \frac{1}{6}D_{c(ab)}V_{cd}\mathcal{B}_d = 0, \\
&\frac{1}{5}A_{abcde}\mathcal{B}_a\mathcal{B}_b\mathcal{B}_c\mathcal{B}_d\mathcal{B}_e + \frac{1}{3}B_{abcd}V_a\mathcal{B}_b\mathcal{B}_c\mathcal{B}_d - \frac{1}{3}B_{bcda}V_a\mathcal{B}_b\mathcal{B}_c\mathcal{B}_d \\
&\qquad - \frac{1}{3}D_{abc}V_{ad}\mathcal{B}_b\mathcal{B}_c\mathcal{B}_d + \frac{2}{3}E_{ab}V_{acd}\mathcal{B}_b\mathcal{B}_c\mathcal{B}_d + \frac{1}{2}D_{abc}V_aV_b\mathcal{B}_c \\
&\qquad - 2E_{ab}V_{ac}V_b\mathcal{B}_c = 0.
\end{aligned} \tag{2.39}$$

Then

$$\Sigma_4 = -\frac{1}{3\sqrt{2}}\left\{B_{bcda}\mathcal{B}_a + 2D_{abc}\mathcal{B}_{ad}\right\}\partial_0\phi_b\partial_0\phi_c\partial_0\phi_d$$
$$-\frac{1}{\sqrt{2}}\left\{\frac{1}{3}B_{abcd}\mathcal{B}_b\mathcal{B}_c\mathcal{B}_d + D_{abc}V_b\mathcal{B}_c + E_{ab}V_{bc}\mathcal{B}_c - E_{cb}V_{ba}\mathcal{B}_c - 2E_{cb}V_b\mathcal{B}_{ac}\right\}\partial_0\phi_a. \tag{2.40}$$

The first equations (2.38) are automatically solved when $\mathcal{B}(\phi)$ has the form given in (1.3), as a consequence of (2.35). The other two equations (2.39) are non-linear in the boundary term and will restrict the A_i parameters in the boundary condition. The result is consistent with what was found already for $a_n^{(1)}$ and $d_5^{(1)}$ from the spin two and three charges. To have a conserved spin four charge for $e_6^{(1)}$ the boundary parameters are restricted to satisfy (1.4), too.

It is worth noting the following. The expressions found for the spin two, three and four charges for the $a_n^{(1)}$ theory on the full line are in agreement with Niedermaier's results in [21]. In [21], expressions are given for the two index tensor that appears in a current of arbitrary spin $s+1$ for $a_n^{(1)}$:

$$\Delta T_{\pm(s+1)} = B_{ab}\partial_\pm^s\phi_a\partial_\pm\phi_b. \tag{2.41}$$

It is found that B_{ab} is restricted to have the form

$$B_{ab} \sim \delta_{a+b,n+1}\Gamma_s^a. \tag{2.42}$$

It seems that this result generalises to other algebras. On the basis of the results presented in this section, one might conjecture

$$B_{ab} \sim \delta_{(a,b)}\frac{\gamma_s^a}{(\gamma_1^a)^s}, \tag{2.43}$$

where γ_s is the s^{th} eigenvector of the Cartan matrix corresponding to the algebra. The symbol $\delta_{(a,b)}$ means $\delta_{a,b}$ if the basis of simple roots is real, as is appropriate for Toda field theories with no mass degenerate conjugate pairs of particles. Otherwise, if the basis is (partly) complex, B_{ab} becomes (partly) anti-diagonal. This happens for $a_n^{(1)}$, where $\delta_{(a,b)} = \delta_{a+b,n+1}$. For $d_{\text{odd}}^{(1)}$ the basis is complex in the $(s,\bar{s})$-subspace. Then $\delta_{(a,b)} = \delta_{a,b}$ for $a,b = 1,\ldots,n-2$, $\delta_{(s,s)} = \delta_{(\bar{s},\bar{s})} = 0$ and $\delta_{s,\bar{s}} = 1$. In fact, one would expect to obtain (2.43) from the quantum result, given that the quantum charges have eigenvalues which can be regarded as the components of the Cartan matrix for g [22] and taking the classical limit. (See ref[23] for perturbative calculations of quantum charges.)

3. A Lax pair for the theory on a half line

To establish notation, the by now standard Lax pair for the affine Toda theory will be written in the form

$$\begin{aligned} a_0 &= H\cdot\partial_1\phi/2+\sum_0^r\sqrt{m_i}(\lambda E_{\alpha_i}-1/\lambda\ E_{-\alpha_i})e^{\alpha_i\cdot\phi/2} \\ a_1 &= H\cdot\partial_0\phi/2+\sum_0^r\sqrt{m_i}(\lambda E_{\alpha_i}+1/\lambda\ E_{-\alpha_i})e^{\alpha_i\cdot\phi/2}, \end{aligned} \tag{3.1}$$

where H, E_{α_i} and $E_{-\alpha_i}$ are the Cartan subalgebra and the generators corresponding to the simple roots, respectively, of a simple Lie algebra of rank r. Included in the set of 'simple' roots is the extra (affine) root, denoted α_0, which satisfies

$$\sum_0^r n_i\alpha_i=0 \qquad n_0=1.$$

The coefficients m_i are related to the n_i by $m_i=n_i\alpha_i^2/8$. The conjugation properties of the generators are chosen so that

$$a_1^\dagger(x,\lambda)=a_1(x,1/\lambda) \qquad a_0^\dagger(x,\lambda)=a_0(x,-1/\lambda). \tag{3.2}$$

Using the Lie algebra relations

$$[H,E_{\pm\alpha_i}]=\pm\,\alpha_i\,E_{\pm\alpha_i} \qquad [E_{\alpha_i},E_{-\alpha_i}]=2\alpha_i\cdot H/(\alpha_i^2),$$

the zero curvature condition for (3.1)

$$\partial_0 a_1-\partial_1 a_0+[a_0,a_1]=0$$

leads to the affine Toda field equations:

$$\partial^2\phi=-\sum_0^r n_i\alpha_i e^{\alpha_i\cdot\phi}. \tag{3.3}$$

What is required is a Lax pair whose zero curvature condition automatically implies the Toda field equations on $x^1<a$ and the boundary condition at $x^1=a$. Merely restricting the old Lax pair to the half line will not do: a better strategy lies in using a Lax pair for the full line together with a 'reflection principle'. A suitable modification of the Lax pair idea will be presented below.

One of the attractive features of the Lax pair representation (3.1) is its rôle in the generation of conserved quantities. The path-ordered exponential

$$U(x_1, x_2; \lambda) = \mathrm{P}\exp \int_{x_1}^{x_2} a_1 dx^1, \tag{3.4}$$

satisfies

$$\frac{d}{dt} U(x_1, x_2; \lambda) = U(x_1, x_2; \lambda) a_0(x_2) - a_0(x_1) U(x_1, x_2; \lambda).$$

Hence, provided the fields ϕ and their derivatives satisfy suitable conditions at $\pm\infty$, the quantity

$$Q(\lambda) = \mathrm{tr}\, U(-\infty, \infty; \lambda) \tag{3.5}$$

will be conserved for any choice of the parameter λ. Indeed, $Q(\lambda)$ provides a generating function for the conservation laws.

Unfortunately, once the field theory is restricted to a half line the same argument cannot be used for the quantity

$$U(-\infty, a; \lambda) = \mathrm{P}\exp \int_{-\infty}^{a} a_1 dx^1 \ . \tag{3.6}$$

On the other hand, consider the path-ordered exponential around the closed contour consisting of the following pieces $H_i\ :\ -\infty < x^1 \leq a;\ x^0 = x^0_i,\ i = 1, 2,\ V_{-\infty}\ :\ x^1 = -\infty;\ x^0_1 \leq x^0 \leq x^0_2$ and $V_a\ :\ x^1 = a;\ x^0_1 \leq x^0 \leq x^0_2$. Since the contour is closed, and since the gauge field a_μ has zero curvature inside the contour, the path-ordered exponential around the contour is unity. That is, explicitly,

$$\mathrm{P}\exp \int_{V_{-\infty}} a_0 dx^0 = \left(\mathrm{P}\exp \int_{H_1} a_1 dx^1\right) \left(\mathrm{P}\exp \int_{V_a} a_0 dx^0\right) \left(\mathrm{P}\exp - \int_{H_2} a_1 dx^1\right). \tag{3.7}$$

Choosing ϕ and its derivatives to vanish as $x^1 \to -\infty$ guarantees the left hand side of (3.7) is unity. If in addition the bondary condition could be used to show that a_0 was a pure gauge at $x^1 = a$, it would be possible to write the middle term in the right hand side of (3.7) as a product of group elements:

$$\mathrm{P}\exp \int_{V_a} a_0 dx^0 = G(x^0_1) G^{-1}(x^0_2).$$

Under these circumstances, eq(3.7) would imply

$$\left(\mathrm{P}\exp \int_{H_2} a_1 dx^1\right) G(x^0_2) = \left(\mathrm{P}\exp \int_{V_{-\infty}} a_0 dx^0\right) G(x^0_1).$$

In other words, the combination

$$\left(\mathrm{P}\exp\int_{-\infty}^{a} a_1 dx^1\right) G(x^0)$$

would be conserved. Unfortunately, the Lax pair is not in a suitable form for the latter part of the argument since it does not take into account the boundary condition at $x^1 = a$. In fact, although this argument is appealing it requires modification to be useful.

To construct a modified Lax pair including the boundary condition, it is first of all convenient to consider an additional special point $x^1 = b\ (> a)$ and two overlapping regions $R_- :\ x^1 \leq (a+b+\epsilon)/2$; and $R_+ :\ x^1 \geq (a+b-\epsilon)/2$. The second region will be regarded as a reflection of the first, in the sense that if $x^1 \in R_+$, then

$$\phi(x^1) \equiv \phi(a+b-x^1). \tag{3.8}$$

The regions overlap in a small interval surrounding the midpoint of $[a,b]$. Then, in the two regions define:

$$\begin{array}{lll} R_- : & \widehat{a}_0 = a_0 - \dfrac{1}{2}\theta(x^1-a)\left(\partial_1\phi + \dfrac{\partial\mathcal{B}}{\partial\phi}\right)\cdot H & \widehat{a}_1 = \theta(a-x^1)a_1 \\ R_+ : & \widehat{a}_0 = a_0 - \dfrac{1}{2}\theta(b-x^1)\left(\partial_1\phi - \dfrac{\partial\mathcal{B}}{\partial\phi}\right)\cdot H & \widehat{a}_1 = \theta(x^1-b)a_1. \end{array} \tag{3.9}$$

Then, it is clear that in the region $x^1 < a$ the Lax pair (3.9) is the same as the old but, at $x^1 = a$ the derivative of the θ function in the zero curvature condition enforces the boundary condition (2.4). Similar statements hold for $x^1 \geq b$ except that the boundary condition at $x^1 = b$ is slightly different in order to accommodate the reflection condition (3.8).

On the other hand, for $x^1 \in R_-$ and $x^1 > a$, $\widehat{a}_1$ vanishes and therefore the zero curvature condition merely implies that $\widehat{a}_0$ is independent of x^1. In turn, this fact implies that ϕ is independent of x^1 in this region. Similar remarks apply to the region $x^1 \in R_+$ and $x^1 < b$. Hence, taking into account the reflection principle (3.8), ϕ is independent of x^1 throughout the interval $[a,b]$, and equal to its value at a or b. For general boundary conditions, a glance at (3.9) reveals that the gauge potential $\widehat{a}_0$ is different in the two regions $R_\pm$. However, to maintain the zero curvature condition over the whole line the values of $\widehat{a}_0$ must be related by a gauge transformation on the overlap. Since $\widehat{a}_0$ is in fact independent of $x^1 \in [a,b]$ on both patches, albeit with a different value on each patch,

the zero curvature condition effectively requires the existence of a gauge transformation $\mathcal{K}$ with the property:

$$\partial_0 \mathcal{K} = \mathcal{K}\, \widehat{a}_0(x^0, b) - \widehat{a}_0(x^0, a)\, \mathcal{K}. \tag{3.10}$$

The group element $\mathcal{K}$ lies in the group G with Lie algebra g, the Lie algebra whose roots define the affine Toda theory.

Next, consider the analogue of (3.5) in the present context:

$$\widehat{Q}(\lambda) = \mathrm{tr}\left(U(-\infty, a; \lambda)\ \mathcal{K}\ U(b, \infty; \lambda)\right). \tag{3.11}$$

Making the usual assumptions concerning the fields at $x^1 = -\infty$ and using (3.10), $\widehat{Q}(\lambda)$ is time-independent. Moreover, the reflection principle (3.8) and conjugation properties (3.2) may be used to rewrite $U(b, \infty; \lambda)$, initially defined as a path-ordered exponential over R_+, as a path-ordered exponential over R_-. Explicitly,

$$U(b, \infty; \lambda) = \left(\mathrm{P}\exp \int_{-\infty}^{a} a_1^{\dagger}(\lambda) dx^1\right)^{\dagger} = \left(\mathrm{P}\exp \int_{-\infty}^{a} a_1(1/\lambda) dx^1\right)^{\dagger}. \tag{3.12}$$

Hence, a more convenient expression for $\widehat{Q}(\lambda)$ is

$$\widehat{Q}(\lambda) = \mathrm{tr}\left(U(-\infty, a; \lambda)\ \mathcal{K}\ U^{\dagger}(-\infty, a; 1/\lambda)\right). \tag{3.13}$$

This definition of $\widehat{Q}(\lambda)$ is reminiscent of formulae introduced previously by Sklyanin and Tarasov [13] for other integrable models, but it is not the same. These authors appear to use the inverse for the second factor, not the hermitian conjugate. However, this choice would not be correct in the present analysis. Indeed, the difference is crucial for Toda field theory.

In order to analyse further the restrictions imposed as a consequence of requiring (3.10), it is useful to make a couple of additional assumptions. In particular, the gauge transformation $\mathcal{K}$ will be assumed to be independent of both x^0 and the fields ϕ or their derivatives. With these assumptions, and using the explicit expressions for $\widehat{a}_0$, eq(3.10) reads,

$$\frac{1}{2}\left[\mathcal{K}(\lambda),\, \frac{\partial \mathcal{B}}{\partial \phi} \cdot H\right]_+ = -\left[\mathcal{K}(\lambda),\, \sum_0^r \sqrt{m_i}(\lambda E_{\alpha_i} - 1/\lambda\ E_{-\alpha_i}) e^{\alpha_i \cdot \phi/2}\right]_-, \tag{3.14}$$

where the field dependent quantities are evaluated at the boundary $x^1 = a$. Eq(3.14) is rather stringent since the boundary term $\mathcal{B}$ does not depend on the spectral parameter λ. Clearly, one solution is always

$$\mathcal{K} = 1, \qquad \mathcal{B} = \text{constant}, \qquad \text{ie } \partial_1\phi\Big|_a = 0. \tag{3.15}$$

To find other solutions, one might begin by noting that if $\mathcal{K}$ solves (3.14) then so does $c\mathcal{K}$, where c is any element of the centre of the group G. Suppose $\mathcal{K}(0)$ exists, then (3.14) implies

$$\left[\mathcal{K}(0), \sum_0^r \sqrt{m_i}\, E_{-\alpha_i}\, e^{\alpha_i\cdot\phi/2}\right]_- = 0,$$

which in turn implies $\mathcal{K}(0)$ is a central element of G (remember, $\mathcal{K}$ has been assumed to be functionally independent of the fields ϕ and therefore $\mathcal{K}(0)$ must commute with each of the generators $E_{-\alpha_i}$). In view of the ambiguity mentioned above, $\mathcal{K}(0)$ may be taken to be unity.

Next, suppose $\mathcal{K}$ has the form

$$\mathcal{K}(\lambda) = \exp\left(\sum_0^\infty k_n\lambda^n\right), \tag{3.16}$$

substitute into (3.14), and solve order by order in λ. The λ^{-1} term is automatic, but order λ^0 yields

$$\frac{\partial\mathcal{B}}{\partial\phi}\cdot H = \left[k_1, \sum_0^r \sqrt{m_i}\, E_{-\alpha_i} e^{\alpha_i\cdot\phi/2}\right]_-, \tag{3.17}$$

requiring k_1 to have the form

$$k_1 = \sum_0^r B_i E_{\alpha_i}, \tag{3.18}$$

and also implying

$$\frac{\partial\mathcal{B}}{\partial\phi} = \sum_0^r B_i\sqrt{\frac{n_i}{2|\alpha_i|^2}}\;\alpha_i\; e^{\alpha_i\cdot\phi/2}.$$

Clearly, the boundary term $\mathcal{B}$ is forced to have the form given in (1.3). Using (3.17), the order λ^1 terms are

$$\left[k_2, \sum_0^r \sqrt{m_i}\, E_{-\alpha_i} e^{\alpha_i\cdot\phi/2}\right]_- = 0,$$

which implies (since k_2 is in the Lie algebra g), $k_2 = 0$. Actually, this should have been expected on noting that if $\mathcal{K}(\lambda)$ solves (3.14), so does $\mathcal{K}^{-1}(-\lambda)$.

The terms of order λ^2 are more interesting and lead to further constraints on the boundary coefficients B_i. Using the previous results, (3.17) and $k_2 = 0$, they may be written:

$$\left[k_3,\, \sum_0^r \sqrt{m_i}\, E_{-\alpha_i} e^{\alpha_i\cdot\phi/2}\right]_- = \frac{1}{12}\left[k_1,\, \left[k_1,\, \frac{\partial \mathcal{B}}{\partial \phi}\cdot H\right]\right]_- + \left[k_1,\, \sum_0^r \sqrt{m_i}\, E_{\alpha_i} e^{\alpha_i\cdot\phi/2}\right]_- . \tag{3.19}$$

Now consider how this equation is graded with respect to the *principal* grading of g. On positive roots, this grading is simply the length of the root; on negative roots it is the negative of this; on the Cartan subalgebra it is zero. The two terms on the right of (3.19) have grade 2 mod h and therefore this must also be the grade of the left hand side. In other words, k_3 must have grade 3 mod h. (Typically $h > 3$; but note, there are algebras, for example a_1 and a_2, which are special cases in having no generators of grade 3.) Therefore, in general,

$$k_3 = \sum_{l(\beta)=3 \bmod h} C_\beta\, E_\beta. \tag{3.20}$$

Using (3.20), (3.18), the expression for the boundary term, and the Lie algebra relations

$$[E_\alpha,\, E_\beta] = \epsilon(\alpha,\beta) E_{\alpha+\beta}, \tag{3.21}$$

the equation corresponding to matching the coefficients of $e^{\alpha_i\cdot\phi/2}$ in eq(3.19) may be written

$$\begin{aligned}\sum_{l(\beta)=3 \bmod h} C_\beta\, \epsilon(\beta,-\alpha_i) E_{\beta-\alpha_i} &= -\frac{1}{12}\sum_{j,k\neq i} B_i B_j B_k (\alpha_i\cdot\alpha_j)\epsilon(\alpha_k,\alpha_j)\sqrt{\frac{n_i}{2|\alpha_i|^2}} E_{\alpha_j+\alpha_k}\\ &+ \sum_j B_j \epsilon(\alpha_i,\alpha_j)\sqrt{\frac{n_i}{2|\alpha_i|^2}}\left(\frac{1}{12}B_i^2\,(\alpha_i^2-\alpha_i\cdot\alpha_j) - \frac{|\alpha_i|^2}{2}\right) E_{\alpha_i+\alpha_j}.\end{aligned} \tag{3.22}$$

There are two cases to consider, the simply-laced *ade* series and the rest.

Simply-laced roots

In this case, consider the generator $E_{\alpha_i+\alpha_j}$. Since no level three root in a simply-laced root system can have the form $\beta = 2\alpha_i + \alpha_j$, (the squared length of such a root would be either 10 or 6), then this generator cannot appear in the sum on the left hand side of (3.22). Since it clearly does not appear in the first term on the right hand side, its coefficient in the second term must vanish, implying restrictions on the constants B_j. Indeed, α_j must be

adjacent to α_i on the (extended) Dynkin-Kac diagram. Therefore, for a particular choice of root, α_i, the coefficients corresponding to its neighbours α_j satisfy

$$\text{either} \qquad B_j = 0 \qquad \text{or} \qquad B_j^2 = 4.$$

Taking each point of the Dynkin-Kac diagram in turn leads to the conclusion that

$$\textbf{either} \quad B_j = 0 \quad \text{for all } j \quad \textbf{or} \quad B_j^2 = 4 \quad \text{for all } j,$$

which translates to precisely the conclusion reached previously, eq(1.4), by considering specific charges of low spin. Eq(3.22) otherwise determines the coefficients C_β in terms of the constants B_j and Lie algebra data.

A complete solution to (3.14) for the a series will be given in an appendix together with a conjecture for d_n, $n > 4$. The latter is based on solving (3.14) completely for $n = 4, 5, 6, 7$. These complete solutions are essentially unique and their existence places no further constraints on the boundary data.

Non simply-laced roots

Here, the story is slightly different and the restrictions on the constants B_j are less stringent. The point is that in these cases, for which there are simple roots of different lengths, $2\alpha_i + \alpha_j$ is also a root provided α_i is a short root adjacent on the Dynkin-Kac diagram to a long root α_j. Hence, if i corresponds to a short root, (3.22) has a term on the left hand side which matches a term on the right for which j is a long neighbour of i; therefore, there is no corresponding constraint equation involving the boundary constants. When α_i and α_j are adjacent but have the same length, the constraints appear as they did for the simply-laced cases. The results obtained by analysing the constraint equations on a case by case basis are reported in an appendix.

4. Integrability

An important aspect of integrability is that the generating functionals of the conserved charges are in involution. In other words, given the canonical equal-time Poisson brackets

$$\{\phi(x,x^0),\,\phi(y,x^0)\} = 0 \quad \left\{\dot\phi(x,x^0),\,\dot\phi(y,x^0)\right\} = 0 \quad \left\{\phi(x,x^0),\,\dot\phi(y,x^0)\right\} = \delta(x-y), \tag{4.1}$$

the charge generating functionals (3.5) satisfy

$$\{Q(\lambda),\, Q(\mu)\} = 0, \tag{4.2}$$

for any choices of λ or μ. As a consequence, the conserved charges themselves are in involution. The crucial step in proving (4.2), relies on establishing the formula

$$\left\{U(\lambda),^{\otimes}\, U(\mu)\right\} = [r(\lambda/\mu),\, U(\lambda)\otimes U(\mu)]_-\,, \tag{4.3}$$

where $U(\lambda)$ represents the path-ordered exponential defined in (3.4). For the details of this, see for example [24,3,2]. In particular, for affine Toda field theory, the classical r-matrix has the form [2]

$$\begin{aligned} r(\lambda/\mu) = {} & \frac{\mu^h + \lambda^h}{\mu^h - \lambda^h}\sum_{i=1}^{r} H_i \otimes H_i \\ & + \frac{2}{\mu^h - \lambda^h}\sum_{\alpha>0}\frac{|\alpha|^2}{2}\left(\lambda^{l(\alpha)}\mu^{h-l(\alpha)}E_\alpha \otimes E_{-\alpha} + \lambda^{h-l(\alpha)}\mu^{l(\alpha)}E_{-\alpha}\otimes E_\alpha\right)\,, \end{aligned} \tag{4.4}$$

where the sum is over all positive roots of g. Notice that

$$r^{\dagger}(\lambda/\mu) = -r(\mu/\lambda), \tag{4.5}$$

a property which will be used below. It is also useful to abbreviate r by writing

$$r(\lambda/\mu) = \sum_i r_i(\lambda/\mu)\, g_i \otimes g_i^{\dagger}, \tag{4.6}$$

where g_i ranges over the generators of the Lie algebra g.

A generating functional for conserved quantities on a half line has been defined in eq(3.13) but it remains to be seen if the corresponding charges are in involution. To investigate this requires an evaluation of the Poisson bracket

$$\left\{U(\lambda)\mathcal{K}(\lambda)U^{\dagger}(1/\lambda),^{\otimes}\, U(\mu)\mathcal{K}(\mu)U^{\dagger}(1/\mu)\right\},$$

using (4.3) repeatedly, where $U(\lambda)$ is to be understood in the sense of (3.13). One obtains a number of terms which are conveniently written:

$$\begin{aligned} \sum_i \Bigg(& r_i(\lambda/\mu)\left(g_i U_\lambda \mathcal{K}(\lambda)U^{\dagger}_{1/\lambda}\otimes g_i^{\dagger}U_\mu\mathcal{K}(\mu)U^{\dagger}_{1/\mu} - U_\lambda g_i\mathcal{K}(\lambda)U^{\dagger}_{1/\lambda}\otimes U_\mu g_i^{\dagger}\mathcal{K}(\mu)U^{\dagger}_{1/\mu}\right) \\ & + r_i(\lambda\mu)\left(g_i U_\lambda\mathcal{K}(\lambda)U^{\dagger}_{1/\lambda}\otimes U_\mu\mathcal{K}(\mu)U^{\dagger}_{1/\mu}g_i - U_\lambda g_i\mathcal{K}(\lambda)U^{\dagger}_{1/\lambda}\otimes U_\mu\mathcal{K}(\mu)g_iU^{\dagger}_{1/\mu}\right) \\ & + r_i(1/\lambda\mu)\left(U_\lambda\mathcal{K}(\lambda)U^{\dagger}_{1/\lambda}g_i^{\dagger}\otimes g_i^{\dagger}U_\mu\mathcal{K}(\mu)U^{\dagger}_{1/\mu} - U_\lambda\mathcal{K}(\lambda)g_i^{\dagger}U^{\dagger}_{1/\lambda}\otimes U_\mu g_i^{\dagger}\mathcal{K}(\mu)U^{\dagger}_{1/\mu}\right) \\ & + r_i(\mu/\lambda)\left(U_\lambda\mathcal{K}(\lambda)U^{\dagger}_{1/\lambda}g_i^{\dagger}\otimes U_\mu\mathcal{K}(\mu)U^{\dagger}_{1/\mu}g_i - U_\lambda\mathcal{K}(\lambda)g_i^{\dagger}U^{\dagger}_{1/\lambda}\otimes U_\mu\mathcal{K}(\mu)g_iU^{\dagger}_{1/\mu}\right)\Bigg). \end{aligned} \tag{4.7}$$

Once the trace is taken in each of the two spaces of the tensor product, and taking into account (4.5), the sum of the first terms in each parenthetic pair exactly cancel. The remaining terms do not automatically sum to zero. However, if it is further assumed, closely following Sklyanin [13], that

$$\Big[r(\lambda/\mu),\, \mathcal{K}^{(1)}(\lambda)\mathcal{K}^{(2)}(\mu)\Big]_- = \mathcal{K}^{(1)}(\lambda)\widetilde{r}(\lambda\mu)\mathcal{K}^{(2)}(\mu) - \mathcal{K}^{(2)}(\mu)\widetilde{r}(\lambda\mu)\mathcal{K}^{(1)}(\lambda), \tag{4.8}$$

then these four terms also cancel. Such an arrangement, not involving U, is not unreasonable given that r and $\mathcal{K}$ are independent of the fields ϕ. In eq(4.8), it was convenient to define

$$\mathcal{K}^{(1)}(\lambda) = \mathcal{K}(\lambda) \otimes 1 \qquad \mathcal{K}^{(2)}(\mu) = 1 \otimes \mathcal{K}(\mu),$$

to facilitate writing the terms on the right hand side, and

$$\widetilde{r}(\lambda\mu) = \sum_i r_i(\lambda\mu) g_i \otimes g_i. \tag{4.9}$$

Notice that the second factor in the tensor product differs from the corresponding factor in (4.6) in not being conjugated. Notice also, that these manipulations work because of the presence of $U^\dagger$ in (3.13), rather than U^{-1}.

These considerations require not only the existence of $\mathcal{K}$ in the sense of (3.10), but also the same assumption as before, namely, that $\mathcal{K}$ is independent of the fields ϕ, and their time derivatives. Otherwise, there would be extra terms in (4.7), since one would have to worry about the Poisson brackets of $\mathcal{K}$ and the other factors in $\widehat{Q}$. On the other hand, it was noted previously that given (3.14), the quantity $\mathcal{K}$ is essentially unique and the boundary conditions on the fields are strongly restricted. There is a danger that eq(4.8) is not compatible with these assumptions. Since (4.8) is bilinear as far as the two $\mathcal{K}$'s are concerned, multiplying $\mathcal{K}$ by a central element of the group G (the only ambiguity in (3.10)), has no effect whatsoever. It therefore needs to be checked that (4.8) places no stronger constraints on the boundary conditions or, otherwise fails to be compatible with $\mathcal{K}$.

In fact, as will be shown below, the solutions already found for $\mathcal{K}$ satisfy (4.8) identically.

In a sense, (3.14) is a more fundamental equation than the defining equation (4.3). The argument could be turned around: given $\mathcal{K}$, eq(4.8) may be regarded as an equation for the classical r-matrix itself. This is reminiscent of an observation made in [20] noting

that once the reflection factors are known in the quantum field theory then the bootstrap relations would imply the S-matrix elements. From this point of view, the reflection factors are fundamental quantities.

$\mathcal{K}$ - r compatibility

First, rewrite (3.14), taking into account the form of the boundary condition and that $\mathcal{K}$ is independent of the fields, to yield (for $i = 0, \ldots, r$)

$$\mathcal{K}(\lambda)\left(B_i\frac{\alpha_i\cdot H}{|\alpha_i|^2}+\lambda E_{\alpha_i}-\frac{1}{\lambda}E_{-\alpha_i}\right)=\left(-B_i\frac{\alpha_i\cdot H}{|\alpha_i|^2}+\lambda E_{\alpha_i}-\frac{1}{\lambda}E_{-\alpha_i}\right)\mathcal{K}(\lambda). \tag{4.10}$$

Define the Lie algebra elements

$$X_i^{\pm}(\lambda)=\pm\lambda\, B_i\frac{\alpha_i\cdot H}{|\alpha_i|^2}+\lambda^2E_{\alpha_i}-E_{-\alpha_i}. \tag{4.11}$$

In terms of these, (4.10) reads

$$K(\lambda)X_i^+(\lambda)K^{-1}(\lambda)=X_i^-(\lambda). \tag{4.12}$$

First, it can be shown that the solution to (4.12) is unique up to multiplication by a central element of G. To demonstrate this, note first that $X_i^{\pm}(\lambda)$ can be used to generate all of the Lie algebra g by repeated commutation. In other words, if one defines

$$X^{\pm}_{ij....k}=[X_i^{\pm},[X_j^{\pm},[......., X_k^{\pm}]]..]$$

then there exists a subset $\bar{X}_i^{\pm}$ of all the $X^{\pm}_{ij...k}$ which span g. Clearly this is true at $\lambda = 0$, since $X_i^-(0) = -E_{-\alpha_i}$. The generators corresponding to the negative simple roots may be used to manufacture the generators for all the negative roots, and then the highest weight generator $E_{-\alpha_0}$ can be used with the negative simple roots to obtain all the positive roots and the Cartan subalgebra. This defines a spanning basis $\bar{X}_i^{\pm}(0)$. Consider the set $\bar{X}_i^{\pm}(\lambda)$. In terms of an orthonormal basis T_i for g,

$$\bar{X}_i^{\pm}(\lambda)=A_{ij}(\lambda)T_j$$

where $A_{ij}(\lambda)$ is a matrix with polynomial entries in λ. The quantities $\bar{X}_i^{\pm}(\lambda)$ will span g if and only if the matrix A_{ij} is invertible. However, the determinant of A_{ij} does not

identically vanish since it is non-zero at $\lambda = 0$, so it must be a non-vanishing polynomial in λ. Hence, except for a finite number of values of λ, $\bar{X}_i^{\pm}(\lambda)$ spans g.

Next, suppose there are two solutions K_1, K_2 to (4.12) . Then

$$(K_2^{-1}K_1)^{-1}X_i^+(\lambda)K_2^{-1}K_1 = K_1^{-1}X_i^-(\lambda)K_1 = X_i^+(\lambda)$$

and the same equation holds for $X^+_{ij....k}$, and in particular for $\bar{X}_i^+$, and so g is fixed under conjugation by $K_2^{-1}K_1$. It follows from Schur's lemma that $K_2^{-1}K_1$ must be in the centre of G.

In order to demonstrate that (4.12) implies (4.8) it is convenient to consider

$$\begin{aligned} L =&(K^{(1)}(\lambda)K^{(2)}(\mu))^{-1}r(\lambda/\mu)K^{(1)}(\lambda)K^{(2)}(\mu) - r(\lambda/\mu) \\ &+ (K^{(1)}(\lambda))^{-1}\tilde{r}(\lambda\mu)K^{(1)}(\lambda) - (K^{(2)}(\mu))^{-1}\tilde{r}(\lambda\mu)K^{(2)}(\mu), \end{aligned}$$

and argue that L vanishes.

As a first step, it may be shown using the explicit epression for the classical r-matrix, eq(4.4), that

$$[L, X_i^-(\lambda)\otimes \mathbf{1} + \mathbf{1}\otimes X_i^-(\mu)] = 0. \tag{4.13}$$

For the next step, consider equation (4.13) at $\lambda = \mu$. Along this line L commutes with all generators of the form $T_i \otimes \mathbf{1} + \mathbf{1} \otimes T_i$. This implies that it must be proportional to the Casimir-like operator $\mathbf{C} = \sum_i T_i \otimes T_i$; that is

$$L(\lambda, \lambda) = \gamma(\lambda)\sum_i T_i \otimes T_i$$

To determine $\gamma(\lambda)$, multiply this equation by $X_i^-(\lambda)\otimes X_j^-(\lambda)$ and take the trace in both first and second space. The left hand side reads

$$\begin{aligned} \mathrm{Tr}_{1,2}\left(X_i^-(\lambda)\otimes X_j^-(\lambda)\,L(\lambda,\lambda)\right) =&\ \lim_{\lambda\to\mu}\mathrm{Tr}_{1,2}\left(X_i^-(\lambda)\otimes X_j^-(\mu)L(\lambda,\mu)\right) \\ =&\ \lim_{\lambda\to\mu}\mathrm{Tr}_{1,2}\left(X_i^+(\lambda)\otimes X_j^+(\mu)r(\lambda/\mu)\right) \\ &- \lim_{\lambda\to\mu}\mathrm{Tr}_{1,2}\left(X_i^-(\lambda)\otimes X_j^-(\mu)r(\lambda/\mu)\right) \\ &+ \lim_{\lambda\to\mu}\mathrm{Tr}_{1,2}\left(X_i^+(\lambda)\otimes X_j^-(\mu)\tilde{r}(\lambda\mu)\right) \\ &- \lim_{\lambda\to\mu}\mathrm{Tr}_{1,2}\left(X_i^-(\lambda)\otimes X_j^+(\mu)\tilde{r}(\lambda\mu)\right) \end{aligned}$$

but, using the cyclic property of trace and explicitly substituting in $r(\lambda/\mu)$ and $\tilde{r}(\lambda\mu)$ the first two terms and last two terms cancel. Therefore, the left hand side vanishes. If

$B_k = 0$ then choosing $i = j = k$ gives a non-vanishing coefficient to $\gamma(\lambda)$, whilst if all the B_i are non-vanishing it can still be arranged for the coefficient to be non-vanishing by choosing i and j to correspond to neighbouring points on the Dynkin diagram. Thus, $L(\lambda,\lambda)$ vanishes.

Finally, the proof may be completed by demonstrating that $L(\lambda,\mu)$ also vanishes away from the line $\lambda = \mu$. To do this, consider (4.13) as an equation for the variable L. The solutions form a vector space with at most dimension one. Suppose there are two linearly independent solutions $S_1(\lambda,\mu),\ S_2(\lambda,\mu) \in g \otimes g$. A linear combination of these, S, may always be found such that $\mathrm{Tr}_{1,2}(S\mathbf{C}) = 0$; for, if

$$\mathrm{Tr}_{1,2}(S_1\mathbf{C}) = s_1(\lambda,\mu)$$
$$\mathrm{Tr}_{1,2}(S_2\mathbf{C}) = s_2(\lambda,\mu)$$

where s_1, s_2 are non-vanishing, simply take $S = s_2 S_1 - s_1 S_2$, while if s_1 vanishes, take $S = S_1$ (or, if s_2 vanishes, take $S = S_2$). Now working in some matrix representation of $g \otimes g$, the entries of S are rational functions of λ and μ, since S is the solution to a set of simultaneous linear equations. So by multiplying by suitable powers of $(\lambda - \mu)$, S can be arranged to be finite and non-zero at generic points on the line $\lambda = \mu$. Along the line $\lambda = \mu$, S should be proportional to $\mathbf{C}$. However, since $\mathrm{Tr}_{1,2}(S\mathbf{C}) = 0$, it would follow that S vanishes, which is a contradiction. If the space of solutions to (4.13) is zero-dimensional away from the line $\lambda = \mu$ it follows that L must vanish away from that line and the result is proved. If it is one dimensional, denote a basis vector $\mathbf{C}(\lambda,\mu)$ and by the above, it can be normalised so that it coincides with $\mathbf{C}$ for $\lambda = \mu$. Therefore

$$L = \gamma(\lambda,\mu)\mathbf{C}(\lambda,\mu).$$

Proceeding similarly, as for the case $\lambda = \mu$, multiply this equation by $X_i^-(\lambda) \otimes X_j^-(\mu)$ and take the trace in both spaces. The left hand side vanishes, while the coefficient of $\gamma(\lambda,\mu)$ is a continous function of λ and μ. For appropriate i and j, it is non-vanishing along the line $\lambda = \mu$ and, by continuity non-vanishing in some neighbourhood of this line. It follows that L must vanish for generic λ and μ.

5. Summary and discussion

In this article much of the detail omitted from [20] has been included. However, the approach advocated there is quite limited since the possibility remains that new conditions might be needed to guarantee the existence of charges of spin greater than four. To compensate for this limitation, a development of the Lax pair idea has been presented which is able to take into account the boundary condition. Within this scheme, in order to make progress, it is nevertheless necessary to make some natural assumptions which lead eventually to the basic equations (3.14) or (4.10). Given these assumptions, the same restrictions on the boundary conditions are obtained in those situations where both approaches are feasible. However, the Lax pair scheme also allows conjectures to be formulated in all other cases.

The set of equations (4.10) are interesting by themselves. Although a number of exact solutions have been found to these equations, for any given Lie algebra the solutions have very little freedom, and indeed there is not yet a proof of existence in the general case. It is intriguing that the solutions to (4.10) are also compatible with (4.8) since this was not guaranteed by the method for constructing $\mathcal{K}$. The relationship between the two quantities r and $\mathcal{K}$, and the defining equations for $\mathcal{K}$, needs to be clarified.

All the considerations of this paper have been entirely classical and a question remains concerning the compatibility of the general boundary conditions and quantum integrability. This is a difficult question to which we hope to return in the future.

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Appendix A.

In this Appendix it is explained how eqs (2.35) are solved to find expressions for $T_{\pm 5}$ for $a_n^{(1)}$, $d_5^{(1)}$ and $e_6^{(1)}$.

It is convenient to take $B_{(abcd)}$ to vanish; a totally symmetric part of B_{abcd} can always be added if it makes the final expressions that we find more compact. After introducing a

notation analogous to eq (2.14) eqs (2.35) imply

$$
\begin{aligned}
D_{ijk} =& 4E_{k(i}C_{j)k}, \\
B_{ijkl} - B_{(jk)li} =& - D_{li(j}C_{k)l} + \frac{1}{2}C_{l(j}D_{k)li} + \frac{1}{2}D_{l(jk)}C_{il} \\
& + 2E_{li}C_{lj}C_{lk} - 2\beta^2 E_{l(j}C_{k)l}C_{il}, \\
A_{ijklm} =& -\frac{1}{3}B_{i(jkl}C_{m)i} + \frac{1}{3}C_{i(j}B_{klm)i} + \frac{1}{3}D_{i(jk}C_l{}^iC_{m)}^{\ i} - \frac{2}{3}E_{i(j}C_k^{\ i}C_l{}^iC_{m)}^{\ i}.
\end{aligned}
\tag{A.1}
$$

Here no summation over repeated indices is intended. Symmetrising the second equation in j, k and l and using the fact that B_{ijkl} is symmetric in its last three indices and that its totally symmetric part vanishes, it is found that

$$
\begin{aligned}
B_{ijkl} =& \frac{1}{4}\Big(-D_{li(j}C_{k)l} + \frac{1}{2}C_{l(j}D_{k)li} + \frac{1}{2}D_{l(jk)}C_{il} + 2E_{li}C_{lj}C_{lk} + \\
& - 2E_{l(j}C_{k)l}C_{il} + (l \to j \to k \to l) + (l \to k \to j \to l)\Big).
\end{aligned}
\tag{A.2}
$$

Hence, D_{ijk}, B_{ijkl} and A_{ijklm} can be calculated once E_{ij} has been found. Before calculating E_{ij} there are a few consistency conditions which follow from eqs (A.1). Using

$$
\sum_{i=0}^{n} n_i \alpha_i^a = 0, \tag{A.3}
$$

one finds

$$
\begin{aligned}
0 =& \sum_{k=0}^{n} n_k D_{ijk} = \sum_{k=0}^{n} 4n_k E_{k(i}C_{j)k}, \\
0 =& \sum_{l=0}^{n} n_l \left(B_{ijkl} - B_{(jk)li}\right) \\
=& \sum_{l=0}^{n} n_l \Big(-D_{li(j}C_{k)l} + \frac{1}{2}C_{l(j}D_{k)li} + \frac{1}{2}D_{l(jk)}C_{il} + 2E_{li}C_{lj}C_{lk} - 2E_{l(j}C_{k)l}C_{il}\Big), \\
0 =& \sum_{i=0}^{n} n_i A_{ijklm} \\
=& \sum_{i=0}^{n} n_i \Big(-\frac{1}{3}B_{i(jkl}C_{m)i} + \frac{1}{3}C_{i(j}B_{klm)i} + \frac{1}{3}D_{i(jk}C_l{}^iC_{m)}^{\ i} - \frac{2}{3}E_{i(j}C_k^{\ i}C_l{}^iC_{m)}^{\ i}\Big).
\end{aligned}
\tag{A.4}
$$

Only relations which are obtained by summing over an index appearing twice on the right hand side in eqs (A.1) will be non-trivial. There are more relations that have to be satisfied.

From eq (A.1) it is clear that D_{ijk} will be automatically symmetric in its first two indices. Eq (A.2) will give a B_{ijkl} which is automatically symmetric in its last three indices, but the extra condition

$$B_{(ijkl)} = 0, \tag{A.5}$$

has to be added separately since this is not automatically implied by eq (A.2). Finally, it is not clear from the last of eq (A.1) that A_{ijklm} is symmetric in all its indices. Hence, this symmetry must be imposed explicitly,

$$A_{ijklm} = A_{(ijklm)}. \tag{A.6}$$

From the first of eqs (A.4) it follows that for $a_n^{(1)}$ the elements of E satisfy $E_{ij} = E_{i+1\ j+1}$ for $i, j = 0, \ldots, n$. For $d_5^{(1)}$, on the other hand, (with 4,5 labelling the simple roots on the fork of the Dynkin diagram) E has to be proportional to

$$E \sim \begin{pmatrix} 0 & 0 & 0 & 0 & +1 & -1 \\ 0 & 0 & 0 & 0 & -1 & +1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & +1 & 0 & 0 & 0 & 0 \\ +1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

while for $e_6^{(1)}$, (with 1,4 and 2,5 labelling simple roots corresponding to spots on the long branches of the Dynkin diagram, working towards the centre, and 6 being the centre spot),

$$E = \begin{pmatrix} 0 & -2(a+b) & +2(a+b) & 0 & -a & +a & 0 \\ +2(a+b) & 0 & -2(a+b) & +a & 0 & -a & 0 \\ -2(a+b) & +2(a+b) & 0 & -a & +a & 0 & 0 \\ 0 & -a & +a & 0 & -b & +b & 0 \\ +a & 0 & -a & +b & 0 & -b & 0 \\ -a & +a & 0 & -b & +b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

with a and b arbitrary constants. From these expressions for E, D can be calculated from the first of eq (A.1). Substituting D and E in the second of eq (A.4) restricts E, and

therefore also D, even more in the case of $a_n^{(1)}$ and $e_6^{(1)}$. For $a_n^{(1)}$,

$$E \sim \begin{pmatrix} 0 & 2(n-3) & 4(n-3) & 4(n-5) & \dots & \dots \\ -2(n-3) & 0 & 2(n-3) & 4(n-3) & \dots & \dots \\ -4(n-3) & -2(n-3) & 0 & 2(n-3) & \dots & \dots \\ -4(n-5) & -4(n-3) & -2(n-3) & 0 & \dots & \dots \\ -4(n-7) & -4(n-5) & -4(n-3) & -2(n-3) & \dots & \dots \\ \vdots & -4(n-7) & -4(n-5) & -4(n-3) & \dots & \dots \\ 4(n-7) & \vdots & -4(n-7) & -4(n-5) & \dots & \dots \\ 4(n-5) & 4(n-7) & \vdots & -4(n-7) & \dots & \dots \\ 4(n-3) & 4(n-5) & 4(n-7) & \vdots & \dots & \dots \\ 2(n-3) & 4(n-3) & 4(n-5) & 4(n-7) & \dots & \dots \end{pmatrix}.$$

Hence $a_3^{(1)}$ has no spin four charge, as was to be expected. For $e_6^{(1)}$, $b=0$ and, up to one overall constant E has now been uniquely determined. The quantities B and A may now be calculated from (A.1) and (A.2) and eqs(A.5) and (A.6) checked. Hence for $a_n^{(1)}$, $d_5^{(1)}$ and $e_6^{(1)}$ there is exactly exactly one current.

At first sight, it appears surprising that this overdetermined set of equations does give a solution in the cases where a spin four charge is expected. This is, however, less remarkable than it seems. Eqs (2.35), from which eqs(A.1) were found, are the conditions under which $\partial_\mp T_{\pm 5} = \partial_\pm \Theta_{\pm 3}$ for some $\Theta_{\pm 3}$. Calculating the left hand side of this equation by substituting the Ansatz for $T_{\pm 5}$ as given in (2.34) yields a total derivative plus terms proportional to $(\partial_\pm\phi)^4$, $(\partial_\pm\phi)^2\partial_\pm^2\phi$ or $(\partial_\pm^2\phi)^2$. All these terms have to vanish and this gives the three equations (2.35). However, these three types of terms can be transformed into each other by adding total derivatives. This means that terms will shift from one equation to another, which in turn introduces arbitrary parameters. The details are omitted for lack of space, but one needs each of the eqs(A.1), (A.2), (A.4), (A.5) and (A.6) to determine first these arbitrary parameters and then E, D, B and A. Only for $a_n^{(1)}$, $d_5^{(1)}$ and $e_6^{(1)}$ is there a non-trivial solution.

Once E_{ij}, D_{ijk}, B_{ijkl} and A_{ijklm} are found, E_{ab}, D_{abc}, B_{abcd} and A_{abcde} are obtained by inverting the transformation (2.14)

$$E_{ab} = \sum_{i,j=1}^{n} \left(\alpha^{-1}\right)_a^{\ i} \left(\alpha^{-1}\right)_b^{\ j} E_{ij}, \tag{A.7}$$

etc. This will lead to expressions for the current $T_{\pm 5}$. For $a_n^{(1)}$ this expression is given in (2.37). A totally symmetric term has been added to B_{abcd} in that case to make the expression more appealing. The same was done in the case of $d_5^{(1)}$ leading to

$$
\begin{aligned}
T_{\pm 5} = & -\frac{1}{4\sqrt{2}} \left\{ [\partial_\pm \phi_1 + \partial_\pm \phi_3]^2 - 2\left(\partial_\pm \phi_2\right)^2 \right\} [\partial_\pm \phi_1 - \partial_\pm \phi_3] \left[\left(\partial_\pm \phi_s\right)^2 - \left(\partial_\pm \phi_{\bar{s}}\right)^2 \right] \\
& + \left[\partial_\pm^2 \phi_s \partial_\pm \phi_s - \partial_\pm^2 \phi_{\bar{s}} \partial_\pm \phi_{\bar{s}} \right] [\partial_\pm \phi_1 - \partial_\pm \phi_3] \, \partial_\pm \phi_2 \\
& + \frac{1}{2\sqrt{2}} \left\{ \left[\partial_\pm^2 \phi_1 + \partial_\pm^2 \phi_3 \right] \partial_\pm \phi_2 - \partial_\pm^2 \phi_2 \left[\partial_\pm \phi_1 + \partial_\pm \phi_3 \right] \right\} \left[\left(\partial_\pm \phi_s\right)^2 - \left(\partial_\pm \phi_{\bar{s}}\right)^2 \right] \\
& - \frac{1}{4} \left[\partial_\pm^2 \phi_{\bar{s}} \partial_\pm \phi_s - \partial_\pm^2 \phi_s \partial_\pm \phi_{\bar{s}} \right] \left[[\partial_\pm \phi_1 + \partial_\pm \phi_3]^2 - 2\left(\partial_\pm \phi_2\right)^2 \right] \\
& - \frac{1}{\sqrt{2}} \left[\partial_\pm^2 \phi_{\bar{s}} \partial_\pm \phi_s - \partial_\pm^2 \phi_s \partial_\pm \phi_{\bar{s}} \right] \left[\left(\partial_\pm \phi_1\right)^2 - \left(\partial_\pm \phi_3\right)^2 \right] \\
& + \left[\left(\partial_\pm^2 \phi_s\right)^2 - \left(\partial_\pm^2 \phi_{\bar{s}}\right)^2 \right] \left\{ [\partial_\pm \phi_1 + \partial_\pm \phi_3] + \frac{1}{\sqrt{2}} [\partial_\pm \phi_1 - \partial_\pm \phi_3] \right\} \\
& + \left[\partial_\pm^2 \phi_1 + \partial_\pm^2 \phi_3 \right] \left[\partial_\pm^2 \phi_s \partial_\pm \phi_s - \partial_\pm^2 \phi_{\bar{s}} \partial_\pm \phi_{\bar{s}} \right] - \sqrt{2} \partial_\pm^2 \phi_2 \left[\partial_\pm^2 \phi_{\bar{s}} \partial_\pm \phi_s - \partial_\pm^2 \phi_s \partial_\pm \phi_{\bar{s}} \right] \\
& + \partial_\pm^4 \phi_s \partial_\pm \phi_{\bar{s}} - \partial_\pm^4 \phi_{\bar{s}} \partial_\pm \phi_s .
\end{aligned}
\tag{A.8}
$$

For $e_6^{(1)}$ the expression is far too long to be reported here.

Appendix B. Explicit expressions for $\mathcal{K}$ in special cases

It is possible to solve (4.10) exactly for the special cases $a_n^{(1)}$ and $d_{4,5,6,7}^{(1)}$, and to conjecture a solution for $d_n^{(1)}$ on the basis of the latter.

It is convenient to work within the smallest representation of a_n ($n+1$-dimensional) and to use a representation for the generators corresponding to the simple roots of the form

$$
\left(E_{\alpha_i}\right)_{jk} = \delta_{j\ i-1} \delta_{k\ i} \qquad i, j, k = 0, 1, 2 \ldots, n \bmod h = n+1.
$$

It is also convenient to set $B_i = 2C_i$, to define $C = \prod_0^r C_i^{n_i}$, to let $l(\alpha)$ denote the level of a root, and to define $l_i(\alpha)$ to be the integer coefficient of the root α_i in the simple root decomposition of α. Then, the solution for $\mathcal{K}$ satisfying

$$
\mathcal{K}^\dagger(\lambda) = \mathcal{K}(1/\lambda),
$$

is

$$
1 + \sum_{\alpha>0} \prod_i C_i^{l_i(\alpha)} \left[\left(\frac{2}{1 + C\lambda^h} \right) (-\lambda)^{l(\alpha)} E_\alpha + \left(\frac{2}{1 + C\lambda^{-h}} \right) (-1/\lambda)^{l(\alpha)} E_{-\alpha} \right]. \tag{B.1}
$$

This case is probably the simplest, in the sense that $\mathcal{K}$ is expressible in terms of Lie algebra generators alone. For that reason, the formula has been established by direct calculation for every n.

For the case of d_n, the situation is somewhat more complicated. There, it is convenient to choose to work in the $2n$-dimensional representation and to set

$$
\begin{aligned}
&(E_{\alpha_i})_{jk} = \delta_{j\ i}\delta_{k\ i+1} + \delta_{j\ n+i+1}\delta_{k\ n+i} \qquad i = 1, 2, \ldots, n-1 \qquad j, k = 1, 2, \ldots, 2n \\
&(E_{\alpha_0})_{jk} = \delta_{j\ n+2}\delta_{k\ 1} + \delta_{j\ n+1}\delta_{k\ 2} \\
&(E_{\alpha_n})_{jk} = \delta_{j\ n-1}\delta_{k\ 2n} + \delta_{j\ n}\delta_{k\ 2n-1},
\end{aligned}
$$

corresponding to the choice of roots:

$$
\alpha_i = e_i - e_{i+1} \qquad i = 1, 2, \ldots, n-1, \qquad \alpha_0 = -(e_1 + e_2), \quad \alpha_n = e_{n-1} + e_n,
$$

where $e_i,\ i = 1, 2, \ldots, n$ are orthonormal vectors.

Then, for d_{even}, $\mathcal{K}$ is conjectured to have the form:

$$
\begin{aligned}
1 + \sum_{\alpha>0} \prod_i C_i^{l_i(\alpha)} & \left[\left(\frac{2}{1 + C\lambda^h} \right) (-\lambda)^{l(\alpha)}\, \widehat{E}_\alpha + \left(\frac{2}{1 + C\lambda^{-h}} \right) (-1/\lambda)^{l(\alpha)}\, \widehat{E}_{-\alpha} \right] \\
& + \frac{4C_n C_{n-1}}{(1 + C\lambda^h)(1 + C\lambda^{-h})} \sum_\beta \lambda^{\widehat{l}(\beta)} C^{l(\beta)/2}\, \widehat{E}_\beta,
\end{aligned}
\tag{B.2}
$$

where the sum over β in the last term refers to the set of vectors $\pm 2e_i$, which are not roots, but expressible in terms of the roots:

$$
\begin{aligned}
&2e_n = \alpha_n - \alpha_{n-1},\ 2e_{n-1} = \alpha_{n-1} + \alpha_n,\ 2e_{n-2} = 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n, \ldots \\
&\qquad\qquad\qquad\qquad \ldots, 2e_1 = 2\alpha_1 + \ldots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n,
\end{aligned}
$$

and the quantity $\widehat{l}(\beta)$ is defined by

$$
\widehat{l}(\beta) = \begin{cases} l(\beta) & \text{if } l(\beta) = 0 \text{ mod } 4 \\ h - l(\beta) & \text{if } l(\beta) = 2 \text{ mod } 4. \end{cases}
$$

The matrices corresponding to these vectors are not Lie algebra generators but have the form

$$
\left(\widehat{E}_{2e_i} \right)_{jk} = \delta_{j\ i}\delta_{k\ n+i} = \left(\widehat{E}_{-2e_i} \right)_{kj} \qquad i = 1, \ldots, n, \quad j, k = 1, \ldots, 2n.
$$

Finally, the other matrices $\widehat{E}_\alpha$ are either generators, or conjugate to generators; in fact, they are given by

$$
\widehat{E}_\alpha = \Omega^{1+l(\alpha)}\, E_\alpha\, \Omega^{-1-l(\alpha)},
$$

where

$$\Omega = \mathrm{diag}(1,1,1,\ldots,1,-1,1,-1,-1,-1,\ldots,-1).$$

However, why this should be the case remains a mystery. The expression has been checked for the cases d_4, d_6 using Mathematica.

For d_4, the following relation also holds

$$\mathcal{K}(\lambda)\mathcal{K}(-\lambda) = \left(\frac{1+C\lambda^6}{1-C\lambda^6}\right)^2 1.$$

An equation of this form was to be expected, replacing $\lambda \to -\lambda$ in (4.10), given the uniqueness of the solution up to a scalar factor but the corresponding relation for all the other cases has not been established.

For d_{odd}, the conjectured solution is a similar expression to (B.2), except for the extra $2n$ terms which have a different, simpler form:

$$\begin{aligned} 1 + \sum_{\alpha>0}\prod_i C_i^{l_i(\alpha)} & \left[\left(\frac{2}{1+C\lambda^h}\right)(-\lambda)^{l(\alpha)}\,\widehat{E}_\alpha + \left(\frac{2}{1+C\lambda^{-h}}\right)(-1/\lambda)^{l(\alpha)}\,\widehat{E}_{-\alpha}\right] \\ & + C_n C_{n-1} \sum_{l(\beta)=2 \bmod 4}\left(\frac{\lambda^{l(\beta)}}{1+C\lambda^h}\widehat{E}_\beta + \frac{\lambda^{-l(\beta)}}{1+C\lambda^{-h}}\widehat{E}_{-\beta}\right). \end{aligned} \tag{B.3}$$

This has been checked explicitly for d_5 and d_7.

Appendix C. The special case $a_2^{(2)}$

It is quite instructive to consider the spin five charges for $a_2^{(2)}$ explicitly.

To facilitate calculation, it is convenient to choose a normalisation for the roots for which the equation of motion is

$$\partial_+\partial_-\phi = -\frac{1}{2}V'(\phi), \qquad V(\phi) = e^{2\phi} + 2e^{-\phi}. \tag{C.1}$$

Then, the appropriate spin ± 6 densities are

$$T_{\pm 6} = (\partial_\pm\phi)^6 - 5(\partial_\pm\phi)^3\partial_\pm^3\phi + 5(\partial_\pm^2\phi)^3 + 3(\partial_\pm^3\phi)^2, \tag{C.2}$$

satisfying

$$\partial_\mp T_{\pm 6} = \partial_\pm \Theta_{\pm 4} \tag{C.3}$$

where

$$\Theta_{\pm 4} = -\frac{1}{8}\left[4(\partial_\pm\phi)^2\partial_\pm^2\phi(-15V' + 6V''') + 12(\partial_\pm^2\phi)^2V'' + (\partial_\pm\phi)^4(10V'' - 6V'''')\right].$$

Insisting the combination $T_6 - T_{-6} + \Theta_4 - \Theta_{-4}$ is a total time derivative in the presence of a boundary term at $x^1 = 0$ requires the boundary term in the lagrangian to have the form

$$\mathcal{B} = A_1 e^{\phi} + A_0 e^{-\phi/2} \tag{C.4}$$

where

$$A_0(A_1^2 - 2) = 0.$$

The above normalisation is less convenient for solving (3.14), however.

In order to discover an expression for $\mathcal{K}$ in this case, it is first necessary to obtain the data for (4.10) by folding $a_2^{(1)}$. Ie, if α_0, α_1 and α_2 are the simple roots of $a_2^{(1)}$, the relevant roots for $a_2^{(2)}$ are $\beta_0 = (\alpha_0 + \alpha_2)/2$ and $\beta_1 = \alpha_1$. It is convenient to take the corresponding generators to be

$$E_{\beta_0} = \sqrt{2}(E_{\alpha_0} + E_{\alpha_2})$$

and to work in the three-dimensional representation of a_2, where

$$\beta_1 \cdot H = \mathrm{diag}(1, -1, 0) \quad \text{and} \quad \beta_0 \cdot H = -\frac{1}{2}\mathrm{diag}(1, -1, 0),$$

and to set $B_0 = -\sqrt{2}C_0,\ B_1 = 2C_1$. Then, (4.10) has a solution, unique up to a scalar factor, provided

$$C_0(C_1^2 - 1) = 0,$$

which is the same condition as the above once the differing normalisations are accounted for. The solution for $\mathcal{K}$ is:

$$(C_1 - \lambda^3)\mathbf{1} +$$
$$\begin{pmatrix} 0 & \frac{-2\lambda(C_1^2 - C_1\lambda^3(1-C_0^2) + C_0^2\lambda^6)}{1+\lambda^6} & -2C_0\lambda^2 \\ \frac{2\lambda^2(C_0^2C_1 + (C_0^2 - C_1^2)\lambda^3 + C_1\lambda^6)}{1+\lambda^6} & 0 & 2C_0C_1\lambda \\ 2C_0C_1\lambda & -2C_0\lambda^2 & \frac{-2\lambda^3(C_0^2 - C_1^2 + C_1(1-C_0^2)\lambda^3)}{1+\lambda^6} \end{pmatrix}$$

For the special case $C_0 = 0$, this simplifies, and is proportional to:

$$\mathbf{1} + \begin{pmatrix} 1 & \frac{-2C_1\lambda}{1+\lambda^6} & 0 \\ \frac{-2C_1\lambda^5}{1+\lambda^6} & 1 & 0 \\ 0 & 0 & \frac{2C_1\lambda^3}{1+\lambda^6} \end{pmatrix}.$$

The latter satisfies

$$\mathcal{K}^\dagger(\lambda) = \mathcal{K}(1/\lambda).$$

Appendix D. The other non simply-laced cases

The implications of eq(3.22) for the non simply-laced root systems have been analysed and are listed in this appendix ($B_i = 2C_i$). The corresponding $\mathcal{K}$-matrices have not been calculated for most of the cases. The simple solution (3.15) is always a possibility and will not be listed separately for each case.

$a_{2n}^{(2)}$ $(n > 2)$:

$$\begin{array}{llll} \textbf{either} & C_i = \pm 1 & 0 \le i \le n-1, & C_n \text{ arbitrary}, \\ \textbf{or} & C_i = 0 & 1 \le i \le n, & C_0 \text{ arbitrary}, \end{array}$$

where C_n is the shortest simple root.

$a_4^{(2)}$:

$$\begin{array}{llll} \textbf{either} & C_0, C_1 = \pm 1, & & C_2 \text{ arbitrary}, \\ \textbf{or} & C_0 = \pm 1, & C_2 = 0, & C_1 \text{ arbitrary}, \\ \textbf{or} & C_1, C_2 = 0, & & C_0 \text{ arbitrary}, \end{array}$$

$b_n^{(1)}$:

$$C_i = \pm 1 \qquad 0 \le i \le n-1, \qquad C_n \text{ arbitrary},$$

where n labels the short simple root.

$a_{2n-1}^{(2)}$:

$$\begin{array}{lll} \textbf{either} & C_i = \pm 1 \quad \text{for all } i, & \\ \textbf{or} & C_i = 0 \quad 0 \le i \le n-1, & C_n \text{ arbitrary}, \end{array}$$

where n labels the long simple root.

$c_n^{(1)}$:

$$\begin{array}{lll} \textbf{either} & C_i = \pm 1 \quad \text{for all } i, & \\ \textbf{or} & C_i = 0 \quad 1 \le i \le n-1, & C_0, C_n \text{ arbitrary}, \end{array}$$

where n labels the long simple root.

$d_n^{(2)}$:

$$C_i = \pm 1 \quad 1 \le i \le n-1, \quad C_0, C_n \text{ arbitrary},$$

where n labels the short simple root.

$g_2^{(1)}$

$$C_0, C_1 = \pm 1, \quad C_2 \text{ arbitrary},$$

where 2 labels the short root.

$d_4^{(3)}$:

$$\begin{array}{ccc} \textbf{either} & C_i = \pm 1 \quad \text{for all } i, & \\ \textbf{or} & C_0, C_1 = 0, & C_2 \quad \text{arbitrary} \end{array},$$

where 2 labels the long simple root.

$f_4^{(1)}$

$$\begin{array}{ccc} \textbf{either} & C_i = \pm 1 \quad \text{for all } i, & \\ \textbf{or} & C_i = \pm 1 \quad 0 \leq i \leq 2, & C_3, C_4 = 0, \end{array}$$

where $3, 4$ label the short simple roots.

$e_6^{(2)}$:

$$\begin{array}{ccc} \textbf{either} & C_i = \pm 1 \quad \text{for all } i, & \\ \textbf{or} & C_i = 0 \quad 0 \leq i \leq 2, & C_3, C_4 = \pm 1, \end{array}$$

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